MAT 216: REAL ANALYSIS I

STREAM: BED (Arts/Science) **DURATION: 3 Hours**

INSTRUCTIONS TO CANDIDATES

i. *Answer ALL Questions from **section A** and any **THREE** from **section B**.*

ii. *Do not write on the question paper.*

SECTION A (31 Marks)

Answer ALL questions in this section

Question One (16 Marks)

a) Suppose $x, y, z \in \mathbb{R}^n$, show that

i) $\| x + y \| \leq \| x \| + \| y \|$

ii) $\| x - z \| \leq \| x - y \| + \| y - z \|$ (4 Marks)

b) Show that a sequence (x_n) converges to zero if and only if the sequence $(|x_n|)$ converges to zero. (5 Marks)

c) Apply the integral test to the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$. (3 Marks)

d) Show that $\lim_{x \to 0} f(x)$ where $f(x) = \frac{|x|}{x} = \begin{cases} 1 & if\ x > 0 \\ -1 & if\ x < 0 \end{cases}$ does not exists.

(4 Marks)

Question Two (15 Marks)

a) Define the following terms

i) Interior point

ii) Boundary point (4 Marks)

b) Let A, B, and C be subsets of a universal set U. show that

i) $A \cap (BUC) = (A \cap B)U(A \cap C)$

ii) $(AUB)^c = A^c \cap B^c$ (4 Marks)

c) Let $x, y, z \in \mathbb{F}$, show that

i) $x + y = x + z$, then $y = z$

ii) $x + y = 0$, then $x = -y$ (3 Marks)

d) Show that $\lim_{n\to\infty} \frac{1}{n} = 0$. (4 Marks)

SECTION B (39 Marks)

Answer any THREE questions from this section.

Question Three (13 Marks)

a) Let (s_n) and (t_n) be sequences of real numbers and let $s \in \mathbb{R}$ if for some positive real number k and some $N_1 \in \mathbb{N}$ we have $|s_n - s| \leq k|t_n|$ for all $n \geq N_1$ and if $\lim_{n\to\infty} t_n = 0$. Show that $\lim_{n\to\infty} s_n = s$. (4 Marks)

b) Let (s_n) be a sequence of real number. If $\lim_{n\to\infty} s_n = l_1$ and $\lim_{n\to\infty} s_n = l_2$, then $l_1 = l_2$. (5 Marks)

c) Show that every convergent sequence of real numbers is bounded. (4 Marks)

Question Four (13 Marks)

a) Let a sequence (s_n) be a bounded sequence. Prove that if (s_n) is monotonically increasing then it converges to its supremum. (4 Marks)

b) Prove that a monotone sequence converges if and only if it is bounded. (4 Marks)

c) Show that $(\frac{n+1}{n})$ is a convergent sequence. (5 Marks)

Question Five (13 Marks)

a) Prove that every convergent sequence (s_n) is a Cauchy sequence. (5 Marks)

b) Let S be a nonempty subset of an ordered field φ and $M \in \varphi$. Prove that $M = supS$ if and only if M is an upper bound for S and for any $\varepsilon \in \varphi$ with $\varepsilon > 0$, there is an element $s \in S$ such that $M - \varepsilon < S$. (8 Marks)

Question Six (13 Marks)

a) Given any two sets A and B, show that if $A = B$ then $(A \subseteq B)\Lambda(B \subseteq A)$. (4 Marks)

b) Let $f: X \mapsto Y$ and $g: Y \mapsto Z$ such that $ran(f) \subseteq dom(g)$, prove that

i) If f and g are onto then so is the composition function gof.

ii) If f and g are one-to-one then so is the composition function gof.

(5 Marks)

c) Prove that if two sets A and B are open then $A \cap B$ is open. (4 Marks)

Question Seven (13 Marks)

a) Let A and B be nonempty subsets of $\mathbb{R}$ which are bounded above. Show that the set $S = \{a + b: a \in A \,\&\, b \in B\}$ is bounded above and $supS = supA + supB$. (5 Marks)

b) Show that the function $f(x) = x$ is uniformly continuous on $\mathbb{R}$. (3 Marks)

c) Given $\lim_{x\to 3}(2x - 5) = 1$, find δ such that $|2x - 5) - 1| < 0.01$ whenever $0 < x - 3 < \delta$. (3 Marks)

d) Define the term supremum. (2 Marks)

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