132

ALUPE UNIVERSITY

OFFICE OF THE DEPUTY VICE CHANCELLOR

ACADEMICS, RESEARCH AND STUDENTS' AFFAIRS

UNIVERSITY EXAMINATIONS

2022/2023 ACADEMIC YEAR

FIRST YEAR FIRST SEMESTER REGULAR MAIN EXAMINATION

FOR THE DEGREE OF BACHELOR OF EDUCATION ARTS/SCIENCE

COURSE CODE: MAT 312

COURSE TITLE: COMPLEX ANALYSIS I

DATE: 16TH DEC 2022 **TIME:** 2.00PM – 5.00PM

INSTRUCTION TO CANDIDATES

- SEE INSIDE

THIS PAPER CONSISTS OF 4 PRINTED PAGES PLEASE TURN OVER

MAT 312: COMPLEX ANALYSIS I

STREAM: BED (Arts/Science) **DURATION: 3 Hours**

INSTRUCTIONS TO CANDIDATES

i. *Answer ALL Questions from **section A** and any **THREE** from **section B**.*
ii. *Do not write on the question paper.*

SECTION A (31 Marks)

Answer ALL questions in this section

Question One (16 Marks)

a) Write the following expressions in the form $x + iy, x, y \in R$:

i) $(3 + 4i)^2$

ii) $\frac{2+3i}{3-4i}$. (6 Marks)

b) Differentiate $f(z) = z^2 + z$ from first principles. (4 Marks)

c) Let $z_n \in \mathbb{C}$. Show that $\sum_{n=0}^{\infty} z_n$ is convergent if, and only if, both $\sum_{n=0}^{\infty} Re(z_n)$ and $\sum_{n=0}^{\infty} Re(z_n)$ are convergent. (6 Marks)

Question Two (15 Marks)

a) Let γ denote the circular path with centre 1 and radius 1, described once anticlockwise and starting at the point 2. Let $f(z) = |z|^2$. Write down a parametrisation of γ. Hence calculate $\int_{\gamma} |z|^2 \ dz$. (5 Marks)

b) Suppose that $f : D \rightarrow \mathbb{C}$ is continuous, $F : D \rightarrow \mathbb{C}$ is an antiderivative of f on D, and γ is a contour from z_0 to z_1. Prove that $\int_{\gamma} f = F(z_1) - F(z_0)$. (4 Marks)

c) Define the following terms

i) Anti-derivative

ii) Continuous function

iii) Open set (6 Marks)

SECTION A (39 Marks)

Answer any THREE questions from this section.

Question Three (13 Marks)

a) Let $z_n \in \mathbb{C}$ and write $z_n = x_n + iy_n, x_n, y_n \in R$. Prove that z_n converges if and only if x_n and y_n converge. (4 Marks)

b) By induction on n, derive De Moivre's Theorem. (4 Marks)

c) Write the function $f(z) = |z|$ in the form $u(x,y) + iv(x,y)$. Using the Cauchy-Riemann equations, decide whether there are any points in $\mathbb{C}$ at which f is differentiable. (5 Marks)

Question Four (13 Marks)

a) Find a Laurent series expansion for

$$f(z) = \frac{1}{z^2 (z - 1)}$$

valid for $0 < |z| < 1$. (4 Marks)

b) Let $f, g : D \to \mathbb{C}$ be holomorphic. Let γ be a smooth path in D starting at z_0 and ending at z_1. Prove the complex analogue of the integration by parts formula: $\int_\gamma fg' = f(z_1)g(z_1) - f(z_0)g(z_0) - \int_\gamma f'g$. (5 Marks)

c) Describe the type of singularity at 0 of $\sin(1/z)$. (4 Marks)

Question Five (13 Marks)

a) Show that for $z, w \in \mathbb{C}$ we have

i) $\cos z = \frac{e^{iz} + e^{-iz}}{2}$

ii) $\sin z = \frac{e^{iz} + e^{-iz}}{2i}$. (4 Marks)

b) Find the radius of convergence of $\sum_{n=1}^{\infty} \frac{2^n z^n}{n}$. (3 Marks)

c) Let $f(z) = 1/(z^2 + z + 1)$ and let $\gamma(t) = 5e^{it}, 0 \le t \le 2\pi$, be the circle of radius 5 centered at 0. Use the Estimation Lemma to bound $\int_\gamma f(z)\, dz$. (6 marks)

Question Six (13 Marks)

a) Let $z, w \in \mathbb{C}$. Show that

i) $z \pm w = \bar{z} \pm \bar{w}$

ii) $\overline{zw} = \bar{z}\bar{w}$ (4 Marks)

b) Let $w_0 \neq 0$ be a complex number such that $|w_0| = r$ and $\arg w_0 = \theta$. Find the polar forms of all the solutions z to $z^n = w_0$, where $n \ge 1$ is a positive integer. (4 Marks)

c) Suppose that z_1 and z_2 are complex numbers, with $z_1 z_2$ real and non-zero. Show that there exists a real number r such that $z_1 = r\bar{z}_2$. (5 marks)

Question Seven (13 Marks)

a) Suppose that $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$, where $z = x + iy$. Use the expressions $x = \frac{z+z}{2}$ and $y = \frac{z-z}{2i}$ to write $f(z)$ in terms of z and simplify the result. (3 Marks)

b) Find all the complex roots of the equation $\cos z = 3$. (5 Marks)

c) Compute the following limits if they exist

i) $\lim_{z \to -i} \frac{iz^3 + 1}{z^2 + 1}$

ii) $\lim_{z \to \infty} \frac{4 + z^2}{(z-1)^2}$. (5 Marks)

****